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Velocity Requirements for Rapid Intercept with Midcourse Corrections

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ABSTRACT

Formulas from Lambert's problem in celestial mechanics are presented for use in a computer program to calculate velocity requirements for rapid intercept of an earth satellite by a rocket fired from the earth or from an orbit about the earth. The rocket is assumed to receive an impulsive velocity at launch and the effect of impulsive midcourse corrections are considered. The effects of the earth rotation are included if the rocket is fired from the earth, but those due to the earth's atmosphere are ignored. The rocket and target satellite are assumed to be moving in conic section in the earth's gravitational field.

Accepted for the Air Force
Joseph R. Waterman, Lt. Col., USAF
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VELOCITY REQUIREMENTS FOR RAPID INTERCEPT WITH MIDCOURSE CORRECTIONS

I. INTRODUCTION

This technical note documents formulas to be used in a Fortran computer program to calculate velocity requirements for rapid intercepts of an earth satellite by a rocket fired from the earth or from an orbit about the earth. The rocket is imagined to receive an impulsive velocity at launch. The effect of impulsive midcourse corrections is considered. The effects of the earth rotation are included if the rocket is fired from the earth, but those due to the earth's atmosphere are ignored. The rocket and the satellite are taken to be moving in conic sections in the $-\mu/r$ gravitational potential of the earth with higher harmonics ignored, where μ is the gravitational constant times the mass of the earth and where r is the distance from the center of the earth. Units used in the program are kilometers and kilometers per second.

Input to the program (with names in the program in capital letters) are:

(a) Calendar date: IMONTH, IDAY, IYEAR

(b) Universal time of rocket firing: IHR, IMIN, SEC

(c) Coordinates of the launching site (if the rocket is launched from the earth)

$$\rho = \text{RADIUS} = \text{distance from center of earth (km)}$$

$$\theta = \text{LONG} = \text{longitude west of Greenwich (deg)}$$

$$\phi = \text{LAT} = \text{north latitude (deg)} \tag{1}$$

(d) Elliptic orbital elements of the target satellite in the coordinate system referred to true equinox and equator of date:

$$a = \text{A} = \text{semi-major axis (km)}$$

$$e = E = \text{eccentricity}$$

$$I = INC = \text{inclination (deg)}$$

$$\Omega = ASC = \text{right ascension of ascending node (deg)}$$

$$\omega = PER = \text{argument of perigee (deg)}$$

$$M_o = ANOM = \text{mean anomaly at time } t_o \text{ of rocket firing} \tag{2}$$

[For the intercept rocket orbit, which could be hyperbolic or parabolic as well as elliptic, we use internally in the program instead of a, M_o the following elements

$$p = \text{semi-latus rectum}$$

$$t_p = \text{time of perigee crossing.} \quad] \tag{3}$$

(e) Similar elliptic orbital elements for inital parking orbit of the rocket if it is launched from orbit.

(f) Increments DVLNCH of magnitude of velocity at launch from an initial possible magnitude VLNCH0 to a final possible magnitude VLNCHl.

(g) Errors in the orbital elements of the target satellite (presumed discovered after launch) if midcourse corrections are considered:

$$\Delta a = DA$$

$$\Delta e = DE$$

$$\Delta I = DINC$$

$$\Delta\Omega = DASC$$

$$\Delta\omega = \text{DPER}$$

$$\Delta M_o = \text{DANOM}$$

(h) Epochs after launch at which midcourse impulsive firings are possible.

(i) Increments of magnitude of midcourse velocity change from an initial possible magnitude of change to a final possible magnitude of change.

Output from the program are:

(a) Minimum time to intercept versus magnitude of impulsive velocity at launch at the given increments of magnitude of velocity at launch;

(b) For each possible epoch of midcourse correction and for each possible intercept orbit from launch to the original target orbit, the minimum time to intercept the new target orbit versus magnitude of impulsive velocity at midcourse correction at the given increments of magnitude of midcourse velocity change.

There will be SC4060 graphical output so that by varying the input one can see the variation in velocity requirements versus time of intercept as a function of target satellite orbit, launching site or parking orbit, and time of launch.

In the following we rigorously derive many standard formulas in the philosophy that too much documentation of a computer program is better than too little.

II. ROTATION OF THE EARTH

Let (x^1, x^2, x^3) be a coordinate system referred to the true equinox and equator of date with origin at the center of the earth. The x^3 axis

points to the north along the axis of rotation of the earth, the x^1 axis lies in the equator and points towards the first point of Aries, and the x^2 axis completes the right hand system.

Let (ρ, θ, ϕ) be the coordinates of the launch site as defined in (1). Let s be the true sidereal time, i.e., the Greenwich hour angle of the first point of Aries. Then the cartesian coordinates of the launch site are

$$x_o^1 = \rho \cos \phi \cos (s - \theta)$$

$$x_o^2 = \rho \cos \phi \sin (s - \theta)$$

$$x_o^3 = \rho \sin \phi \tag{4}$$

The velocity of the rocket as it sits at the launch site due to the rotation of the earth is

$$\dot{x}_o^1 = - x^2 \frac{ds}{dt}$$

$$\dot{x}_o^2 = x^1 \frac{ds}{dt}$$

$$\dot{x}_o^3 = 0 \tag{5}$$

Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be unit vectors in the (x^1, x^2, x^3) coordinate directions. The vector from the center of the earth to the launch site is

$$\vec{r}_o = x_o^1 \vec{e}_1 + x_o^2 \vec{e}_2 + x_o^3 \vec{e}_3 \tag{6}$$

and the velocity of the rocket at the launch site due to the rotation of the earth is

$$\vec{v}_o = \dot{x}_o^1 \vec{e}_1 + \dot{x}_o^2 \vec{e}_2 \tag{7}$$

How is the sidereal time calculated for the given calendar date and universal time UT of launch? First we determine the Julian Day Number JD from the given month, day and year of launch as given in Ref. 1, Table I, p. 445. A computer subroutine can easily be written to do this using the fact that there are 365 days in a year except 366 days in years divisible by 4 . The Julian Date at midnight beginning of day is then JD − 0.5 . The Greenwich mean sidereal time s_o at 0^h universal time on the day of interest is

$$s_o = 6^h 38^m 45^s.836 + 8,640,184^s.542T + 0^s.0929T^2 \qquad (8)$$

where T denotes the number of Julian centuries of 36525 days which, at midnight beginning of day, have elapsed since mean noon on 1900 January 0 at the Greenwich meridian (Julian Date 2415020.0) ; see Ref. 1, p. 474. The Greenwich true sidereal time at a given instant UT of universal time on that day is then

$$s = s_o + \frac{ds}{dt} \times UT + \Delta\psi \cos \epsilon \qquad (9)$$

where

$$\frac{ds}{dt} = (1.002737909265 + 0.589 \times 10^{-10}T)$$

sidereal time seconds per universal time second (10)

and where $\Delta\psi$ is the nutation in longitude and ϵ the obliquity of the ecliptic; see Ref. 2, pp. 75-76. We shall ignore the $\Delta\psi \cos \epsilon$ term, whose largest magnitude is $1^s.3$. This has the effect of changing the launch site longitude by this amount. For use in (5), formula (10) for ds/dt must be multiplied by $2\pi/86,400$ to convert to radians per second.

III. POSITION AND VELOCITY IN A CONIC SECTION ORBIT GIVEN THE ORBITAL ELEMENTS

The equations of motion of a body of negligably small mass in the gravitational field of a spherically symmetric earth are

$$\frac{d^2\vec{r}}{dt^2} = -\frac{\mu\vec{r}}{r^3} \tag{11}$$

where $\mu > 0$ is the gravitational constant times the mass of the earth, $\vec{r}$ is the position vector of the body and $r = |\vec{r}| = (\vec{r} \cdot \vec{r})^{1/2}$. Taking the dot product of both sides of (11) with $d\vec{r}/dt$ we obtain

$$\frac{d^2\vec{r}}{dt^2} \cdot \frac{d\vec{r}}{dt} = -\frac{\mu}{r^3}\,\vec{r} \cdot \frac{d\vec{r}}{dt}$$

which implies

$$\frac{1}{2}\frac{d}{dt}\left(\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}\right) = -\frac{1}{2}\frac{\mu}{r^3}\frac{d(\vec{r} \cdot \vec{r})}{dt} = \mu\frac{d}{dt}\left(\frac{1}{r}\right)$$

Thus we obtain the conservation of energy result

$$v^2 = 2\left(\frac{\mu}{r} + C\right) \tag{12}$$

where $\vec{v} = d\vec{r}/dt$ and C is a constant of integration. Taking the cross product of (11) with $\vec{r}$ we obtain

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{0}$$

which implies

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{0}$$

Thus we obtain the conservation of angular momentum result

$$\vec{r} \times \frac{d\vec{r}}{dt} = \vec{G} \tag{13}$$

where $\vec{G}$ is a constant vector of integration. Taking the dot product of (13) with $\vec{r}$ gives

$$\vec{G} \cdot \vec{r} = 0 \tag{14}$$

which implies that the motion is in a plane perpendicular to $\vec{G}$ and that there are only two independent components of $\vec{G}$, not three.

For the moment we exclude the straight line case and assume $\vec{G} \neq \vec{0}$. Referring to Fig. 1, the values of the ascending node Ω and inclination I of the orbital plane normal to $\vec{G}$ are given by

$$\cos I = \frac{\vec{G} \cdot \vec{e}_3}{G} \qquad 0 \leq I \leq 180^\circ \tag{15}$$

$$\left.\begin{aligned} \cos \Omega &= -\frac{\vec{G}_p \cdot \vec{e}_2}{G_p} \\ \sin \Omega &= \frac{\vec{G}_p \cdot \vec{e}_1}{G_p} \end{aligned}\right\} \qquad 0 \leq \Omega < 360^\circ \tag{16}$$

where $G = |\vec{G}|$ and where $\vec{G}_p$ is the projector of $\vec{G}$ onto the (x^1, x^2) plane,

$$\vec{G}_p = \vec{G} - (\vec{G} \cdot \vec{e}_3)\, \vec{e}_3 \tag{17}$$

Let (u^1, u^2) be the coordinates of the body in a coordinate system with u^1 axis being the intersection of the orbit plane with the (x^1, x^2) plane and with u^2 axis in the orbit plane completing the right hand system.

We define circular coordinates $(r, \tilde{\psi})$ in the orbit plane by

$$\begin{aligned} u^1 &= r \cos \tilde{\psi} \\ u^2 &= r \sin \tilde{\psi} \end{aligned} \tag{18}$$

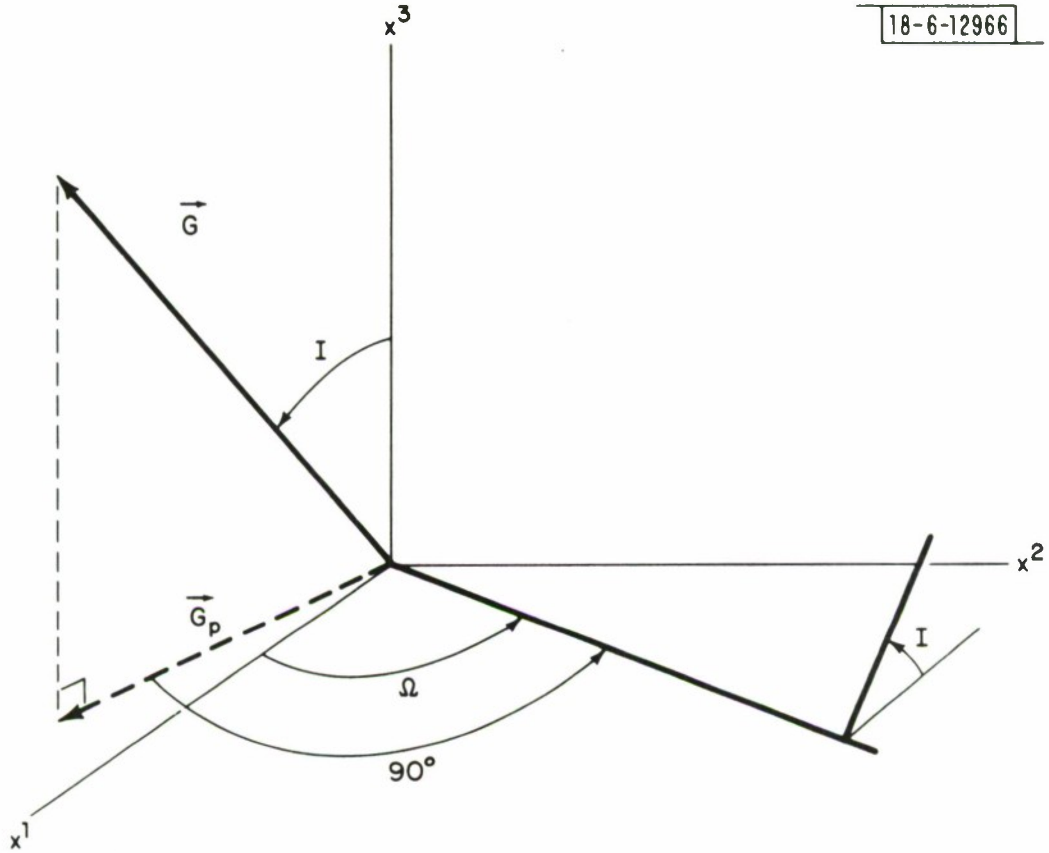

Fig. 1. Relation between inclination I and ascending node Ω of a plane and the normal $\vec{G}$ to the plane.

Then (13) implies

$$r^2 \frac{d\tilde{\psi}}{dt} = G \tag{19}$$

and (12) implies

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\tilde{\psi}}{dt}\right)^2 = 2\left(\frac{\mu}{r} + C\right) \tag{20}$$

Since

$$\frac{dr}{dt} = \frac{dr}{d\tilde{\psi}} \frac{d\tilde{\psi}}{dt} = \frac{dr}{d\tilde{\psi}} \frac{G}{r^2} = -\frac{d}{d\tilde{\psi}}\left(\frac{G}{r}\right)$$

we have

$$\left(\frac{d}{d\tilde{\psi}}\left(\frac{G}{r}\right)\right)^2 + \left(\frac{G}{r}\right)^2 = 2\left(\frac{\mu}{r} + C\right)$$

which implies

$$\pm \frac{d}{d\tilde{\psi}}\left(\frac{G}{r}\right) = \sqrt{-\frac{G^2}{r^2} + 2\left(\frac{\mu}{r} + C\right)}$$

Let

$$q = -\frac{\mu}{G} + \frac{G}{r}$$

$$Q^2 = 2C + \frac{\mu^2}{G^2}$$

We then have

$$\pm \frac{dq}{d\tilde{\psi}} = \sqrt{Q^2 - q^2}$$

which implies

$$\pm (\tilde{\psi} - \omega) = \arccos\left(\frac{q}{Q}\right)$$

where ω is a constant of integration. Since $\cos(+\theta) = \cos(-\theta)$ we have

$$\frac{1}{r} = \frac{\mu}{G^2} \pm \frac{1}{G}\sqrt{2C + \frac{\mu^2}{G^2}} \cos(\tilde{\psi} - \omega)$$

We can have the same curve with a $-$ sign as with a $+$ sign by changing the constant of integration ω by π. Therefore,

$$\frac{p}{r} = 1 + e \cos \psi \tag{21}$$

where

$$p = \frac{G^2}{\mu} > 0 \quad \text{(semi-latus rectum)} \tag{22}$$

$$e = \sqrt{\frac{2G^2 C}{\mu^2} + 1} \geq 0 \quad \text{(eccentricity)} \tag{23}$$

$$\omega = \text{argument of perigee} \tag{24}$$

$$\psi = \tilde{\psi} - \omega = \text{(true anomaly)} \tag{25}$$

We have that $\psi = 0$ at perigee, the point of closest approach to the earth.

The curve represented by (21) is one of the following types:

$$\begin{array}{ll} 0 \leq e < 1 & \text{ellipse} \\ e = 1 & \text{parabola} \\ e > 1 & \text{hyperbola} \end{array} \tag{26}$$

with a focus at the origin.

We define the semi-major axis a by

$$a = \frac{p}{1 - e^2} \tag{27}$$

where

$$a > 0 \quad \text{ellipse}$$

$$a = \infty \quad \text{parabola}$$

$$a < 0 \quad \text{hyperbola} \tag{28}$$

From equations (12), (22), (23) and (27) we have

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{29}$$

Let (y^1, y^2, y^3) be a coordinate system with origin at the center of the earth with y^1 axis pointing towards perigee, with y^3 axis normal to the orbital plane along the angular momentum vector $\vec{G}$, and with y^2 axis completing the right hand system (see Fig. 2). Then we have

$$x^j = \sum_{k=1}^{3} b_{jk} y^k \quad , \quad j = 1,2,3$$

$$y^k = \sum_{j=1}^{3} b_{jk} x^j \quad , \quad k = 1,2,3 \tag{30}$$

where the orthogonal matrix (b_{jk}) is given by

$$b_{11} = \cos\Omega \cos\omega - \sin\Omega \sin\omega \cos I$$

$$b_{12} = -\cos\Omega \sin\omega - \sin\Omega \cos\omega \cos I$$

$$b_{13} = \sin\Omega \sin I$$

$$b_{21} = \sin\Omega \cos\omega + \cos\Omega \sin\omega \cos I$$

$$b_{22} = -\sin\Omega \sin\omega + \cos\Omega \cos\omega \cos I$$

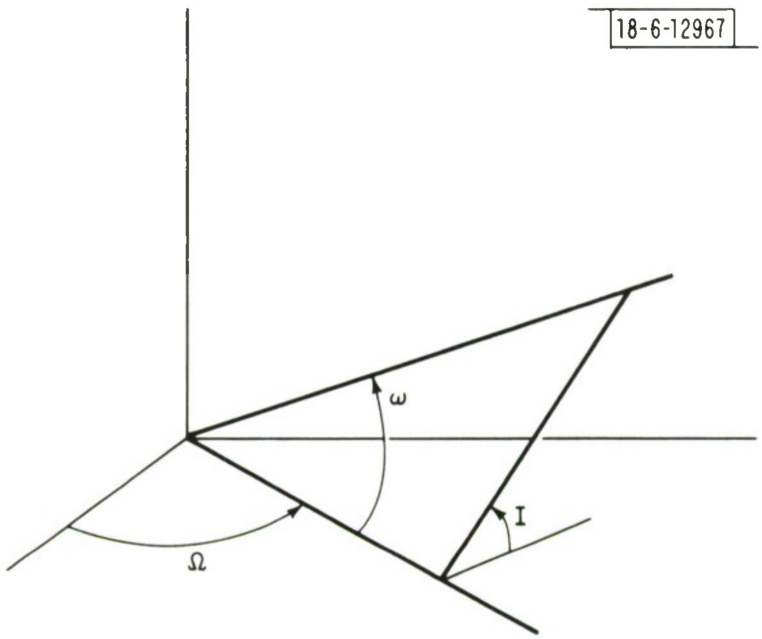

Fig. 2. Euler angles I, Ω, ω.

$$b_{23} = -\cos\Omega \sin I$$

$$b_{31} = \sin\omega \sin I$$

$$b_{32} = \cos\omega \sin I$$

$$b_{33} = \cos I \tag{31}$$

(See Ref. 3, p. 328.) If (y^1, y^2, y^3) is a point in the orbit of the body, $y^3 = 0$ and

$$y^1 = r\cos\psi$$

$$y^2 = r\sin\psi \tag{32}$$

By (19) - (25) and (29) we have

$$\left(\frac{dr}{dt}\right)^2 + \frac{\mu p}{r^2} = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{33}$$

$$r^2 \frac{d\psi}{dt} = \sqrt{\mu p} \tag{34}$$

$$r = \frac{p}{1 + e\cos\psi} \tag{35}$$

III-A. Parabolic Motion

In the case of parabolic motion, $e = 1$, $a = \infty$ and by (34) and (35)

$$\mu^{1/2} p^{-3/2} \int dt = \int \frac{d\psi}{(1+\cos\psi)^2} = \frac{1}{2}\left(\tan\frac{\psi}{2} + \frac{1}{3}\tan^3\frac{\psi}{2}\right)$$

Let t_p be the time of perigee passage when $\psi = 0$. We define the mean motion n and mean anomaly M at time t by

$$n = \mu^{1/2} p^{-3/2} \tag{36}$$

$$M = n(t - t_p) \tag{37}$$

Then the true anomaly ψ at time t is determined by solving the equation

$$2M = \tan\frac{\psi}{2} + \frac{1}{3}\tan^3\frac{\psi}{2} \tag{38}$$

as follows (see Ref. 4, p. 26). Consider the identity

$$\frac{1}{3}\left(\lambda^3 - \frac{1}{\lambda^3}\right) = \left(\lambda - \frac{1}{\lambda}\right) + \frac{1}{3}\left(\lambda - \frac{1}{\lambda}\right)^3$$

with

$$\tan\frac{\psi}{2} = \lambda - \frac{1}{\lambda}$$

$$6M = \lambda^3 - \frac{1}{\lambda^3}$$

Let

$$\lambda = -\tan\gamma$$

$$\lambda^3 = -\tan\beta$$

Then

$$\beta = \frac{1}{2}\cot^{-1}(3M) \qquad 0 \le \beta \le \frac{\pi}{2}$$

$$\gamma = \tan^{-1}\left[(\tan\beta)^{1/3}\right] \qquad -\frac{\pi}{2} \le \gamma \le \frac{\pi}{2}$$

$$\psi = 2\tan^{-1}(2\cot 2\gamma) \qquad -\pi \le \psi \le \pi \tag{39}$$

Given the orbital elements $(p, \ e = 1, \ I, \ \Omega, \ \omega, \ t_p)$ we determine the position and velocity $(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3)$ at time t by first determining ψ from (37) and (39). We determine r by (35) and (y^1, y^2) by (32). We then apply (30) with $y^3 = 0$ to determine the position coordinates. The velocity is given by

$$\dot{x}^j = \sum_{k=1}^{2} b_{jk}\dot{y}^k \quad , \quad j = 1, 2, 3 \tag{40}$$

where

$$\begin{aligned} \dot{y}^1 &= \frac{dr}{dt}\cos\psi - r\sin\psi\frac{d\psi}{dt} \\ \dot{y}^2 &= \frac{dr}{dt}\sin\psi + r\cos\psi\frac{d\psi}{dt} \end{aligned} \tag{41}$$

with dr/dt and $d\psi/dt$ being determined by (33) and (34). $dr/dt < 0$ if $t < t_p$ and $dr/dt > 0$ if $t > t_p$.

III-B. Elliptic Motion

We now suppose that $0 \le e < 1$ and $a > 0$. Equations (29) and (30) imply

$$\pm\, na\int dt = \int\frac{r\,dr}{\sqrt{a^2 e^2 - (r-a)^2}} \tag{42}$$

where the mean motion n is defined by

$$n = \mu^{1/2} a^{-3/2} \tag{43}$$

Let t_p be the time of perigee passage and let the mean anomaly M at time t be given by

$$M = n(t - t_p) \tag{44}$$

In terms of the mean anomaly M_o at time t_o we have

$$M = M_o + n(t - t_o) \tag{45}$$

We define the eccentric anomaly ξ by

$$r = a(1 - e \cos \xi) \tag{46}$$

with $\xi = 0$ when $t = t_p$. We have

$$dr = a\, e \sin \xi \, d\xi$$

and (42) becomes

$$M = n(t - t_p) = \xi - e \sin \xi \tag{47}$$

The $\pm$ sign is eliminated by changing the sign of ξ if necessary without changing (46). See Ref. 5, p. 22.

Equations (35) and (45) give

$$\cos \psi = \frac{\cos \xi - e}{1 - e \cos \xi} \tag{48}$$

which implies

$$1 + \cos \psi = \frac{(1 - e)(1 + \cos \xi)}{1 - e \cos \xi}$$

$$1 - \cos \psi = \frac{(1 + e)(1 - \cos \xi)}{1 - e \cos \xi}$$

These equations may be written

$$2 \cos^2 \frac{\psi}{2} = \frac{1 - e}{1 - e \cos \xi} \cdot 2 \cos^2 \frac{\xi}{2}$$

$$2 \sin^2 \frac{\psi}{2} = \frac{1 + e}{1 - e \cos \xi} \cdot 2 \sin^2 \frac{\xi}{2}$$

Dividing the second equation by the first and taking square roots we obtain

$$\tan \frac{\psi}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{\xi}{2} \tag{49}$$

We have $M = \xi = \psi$ when $\psi = 0 + 2k\pi$ or $\psi = \pi + 2k\pi$, k any integer. Further, modulo 2π we have

$$0 < \psi < \pi \text{ implies } 0 < \xi < \pi \quad , \quad 0 < M < \pi$$

$$\pi < \psi < 2\pi \text{ implies } \pi < \xi < 2\pi \quad , \quad \pi < M < 2\pi \tag{50}$$

which removes any quadrant ambiguity in determining ψ from ξ using (49). The body completes one orbit when M increases by 2π, so that

$$\text{orbital period} = \frac{2\pi}{n} \tag{51}$$

Equations (32), (46) and (48) imply

$$y^1 = r \cos \psi = a(\cos \xi - e) \tag{52}$$

By (46) and (52)

$$\begin{aligned} r^2 \sin^2 \psi = r^2 - r^2 \cos^2 \psi &= a^2 \Big(1 - 2 e \cos \xi + e^2 \cos^2 \xi - \cos^2 \xi \\ &\qquad + 2 e \cos \xi - e^2\Big) \\ &= a^2 (1 - e^2) \sin^2 \xi \end{aligned}$$

so that

$$y^2 = r \sin \psi = a \sqrt{1 - e^2} \sin \xi \tag{53}$$

because of the relations (50).

Given the orbital elements $(p,\ 0 \le e < 1\ ,\ I,\ \Omega,\ \omega,\ t_p)$ we determine the position and velocity $(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3)$ at time t by first determining M from (44) and reducing it to be between 0 and 2π. Then Kepler's equation (47) is solved for ξ using a Newton-Raphson iteration. Namely, if $f(\xi)$ is a function and we wish to find ξ such that $f(\xi) = M$ starting from a nearby value ξ_o, we would define as in Fig. 3

$$M_o = f(\xi_o)$$

$$\Delta\xi_o = \frac{M - M_o}{f'(\xi_o)}$$

$$\xi_1 = \xi_o + \Delta\xi_o \tag{54}$$

continuing the Newton-Raphson iteration to define M_1, $\Delta\xi_1$, ξ_2, etc., until we converge to ξ. Convergence will occur unless $f''(\xi) = 0$. In the specific case of Kepler's equation (47) we take as a first approximation

$$\xi_o = M$$

If $e = 0$, nothing further is required. Otherwise, as in Ref. 5, p. 84, we let

$$M_o = \xi_o - e \sin \xi_o$$

$$\Delta\xi_o = \frac{M - M_o}{1 - e \cos \xi_o}$$

$$\xi_1 = \xi_o + \Delta\xi_o \qquad \text{(modulo } 2\pi) \tag{55}$$

continuing the iteration to get M_1, $\Delta\xi_1$, ξ_2, etc., until we obtain a value ξ_{k+1} such that $|\Delta\xi_k| < \epsilon$, where ϵ is an accuracy constant depending on the number of places in floating point computations on the

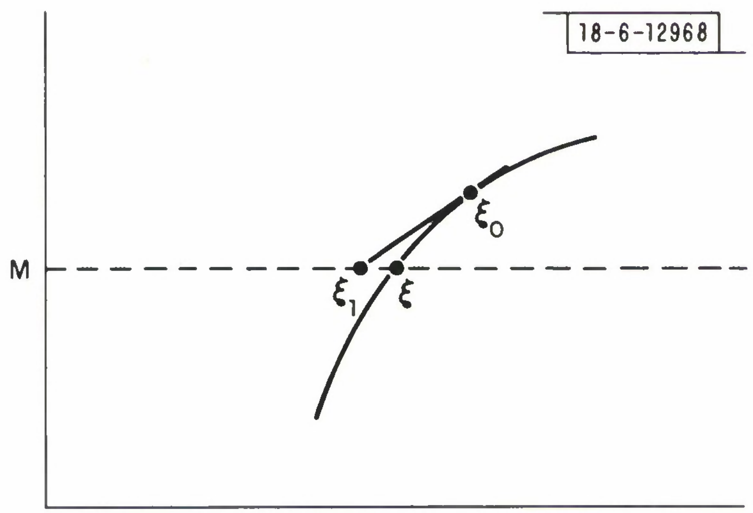

Fig. 3. Newton-Raphson technique.

computer being used. We have

$$f''(\xi) = e \sin \xi$$

which is zero only at $\xi = 0$ and $\xi = \pi$. Thus the Newton-Raphson iteration will converge everywhere except at these two points. But the iteration is not needed at these two points, because the first guess $\xi_o = M$ already gives the correct value of ξ.

Having obtained ξ we evaluate (y^1, y^2) from (52) and (53) and determine (x^1, x^2, x^3) from (30) with $y^3 = 0$.

Differentiating (47) with respect to t we obtain

$$\frac{d\xi}{dt} = \frac{na}{r} \tag{56}$$

so that differentiating (52) and (53) we obtain

$$\dot{y}^1 = -\frac{na^2}{r} \sin \xi \tag{57}$$

$$\dot{y}^2 = \frac{na^2\sqrt{1 - e^2}}{r} \cos \xi \tag{58}$$

The velocity $(\dot{x}^1, \dot{x}^2, \dot{x}^3)$ is then determined by (40).

III-C. Hyperbolic Motion

We now suppose that $e > 1$ and $a < 0$. Equations (27) and (33) imply

$$\pm n|a| \int dt = \int \frac{rdr}{\sqrt{(r + |a|)^2 - |a|^2 e^2}} \tag{59}$$

where the mean motion n is defined by

$$n = \mu^{1/2} \; |a|^{-3/2} \tag{60}$$

Let t_p be the time of perigee passage and let the mean anomaly M at time t be given by

$$M = n(t - t_p) \tag{61}$$

We define the eccentric anomaly ξ by

$$r = |a|(e \cosh \xi - 1) \tag{62}$$

with $\xi = 0$ when $t = t_p$. We have

$$dr = |a| e \sinh \xi \, d\xi$$

and (59) becomes

$$M = n(t - t_p) = e \sinh \xi - \xi \tag{63}$$

The $\pm$ sign is eliminated by changing the sign of ξ if necessary without changing (62). See Ref. 4, p. 27.

Equations (27), (35) and (67) give

$$\cos \psi = \frac{e - \cosh \xi}{e \cosh \xi - 1} \tag{64}$$

which implies

$$1 + \cos \psi = \frac{(e - 1)(1 + \cosh \xi)}{e \cosh \xi - 1}$$

$$1 - \cos \psi = \frac{(e + 1)(\cosh \xi - 1)}{e \cosh \xi - 1}$$

These equations may be written

$$2 \cos^2 \frac{\psi}{2} = \frac{e - 1}{e \cosh \xi - 1} \cdot 2 \cosh^2 \frac{\xi}{2}$$

$$2 \sin^2 \frac{\psi}{2} = \frac{e + 1}{e \cosh \xi - 1} \cdot 2 \sinh^2 \frac{\xi}{2}$$

Dividing the second equation by the first and taking square roots we obtain

$$\tan \frac{\psi}{2} = \sqrt{\frac{e + 1}{e - 1}} \tanh \frac{\xi}{2} \tag{65}$$

We have $M = \xi = \psi$ when $\psi = 0$. Further

$$0 < \psi < \pi \text{ implies } 0 < \xi < \infty , \quad 0 < M < \infty$$

$$-\pi < \psi < 0 \text{ implies } -\infty < \xi < 0 , \quad -\infty < M < 0 \tag{66}$$

which removes any quadrant ambiguity in determining ψ from ξ using (65).

Equations (32), (62) and (64) implies

$$y^1 = r \cos \psi = |a|(e - \cosh \xi) \tag{67}$$

By (62) and (67)

$$\begin{aligned} r^2 \sin^2 \psi = r^2 - r^2 \cos^2 \psi &= |a|^2 (e^2 \cosh^2 \xi - 2e \cosh \xi + 1 - e^2 \\ &\quad + 2e \cosh \xi - \cosh^2 \xi) \\ &= |a|^2 (e^2 - 1) \sinh^2 \xi \end{aligned}$$

so that

$$y^2 = r \sin \psi = |a| \sqrt{e^2 - 1} \sinh \xi \tag{68}$$

because of the relations (66).

Given the orbital elements $(p, \ e > 1, \ I, \ \Omega, \ \omega, \ t_p)$ we determine the position and velocity $(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3)$ at time t by first determining M from (61). Then (63) is solved for ξ using a Newton-Raphson iteration (54). Namely, we take as a first approximation

$$\xi_o = M \quad \text{if} \quad |M| \leq 1$$

$$\xi_o = \sinh^{-1}(M/e) \quad \text{if} \quad |M| > 1$$

Then we let

$$M_o = e \sinh \xi_o - \xi$$

$$\Delta\xi_o = \frac{M - M_o}{e \cosh \xi_o - 1}$$

$$\xi_1 = \xi_o + \Delta\xi_o \tag{69}$$

continuing the iteration to get M_1, $\Delta\xi_1$, ξ_2, etc., until we obtain a value ξ_{k+1} such that

$$|\Delta\xi_k| < \varepsilon \quad \text{if} \quad |\xi_{k+1}| \leq 1$$

$$\frac{|\Delta\xi_k|}{|\xi_{k+1}|} < \varepsilon \quad \text{if} \quad |\xi_{k+1}| > 1$$

where ε is an accuracy constant depending on the number of places in floating point computation on the computer being used. We have

$$f''(\xi) = e \sinh \xi$$

which is zero only at $\xi = 0$. Thus the Newton-Raphson iteration will converge everywhere except at this point. But the iteration is not needed at this point, because the first guess $\xi_o = M$ already gives the correct value of ξ .

Having obtained ξ, we evaluate (y^1, y^2) from (67) and (68) and determine (x^1, x^2, x^3) from (30) with $y^3 = 0$.

Differentiating (63) with respect to t we obtain

$$\frac{d\xi}{dt} = \frac{n|a|}{r} \tag{70}$$

so that differentiating (67) and (68) we obtain

$$\dot{y}^1 = -\frac{n|a|^2}{r} \sinh \xi \tag{71}$$

$$\dot{y}^2 = \frac{n|a|^2 \sqrt{e^2 - 1}}{r} \cosh \xi \tag{72}$$

The velocity $(\dot{x}^1, \dot{x}^2, \dot{x}^3)$ is then determined by (40).

III-D. Straight Line Motion

In the case that $\vec{G} = \vec{0}$ (see equation (13)), the vectors $\vec{r}$ and $\vec{v}$ are always parallel. Thus we have

$$\vec{v} = \frac{dr}{dt} \frac{\vec{r}}{r}$$

and (12) implies

$$v = \frac{dr}{dt} = \pm\sqrt{2(\frac{\mu}{r} + C)} \tag{73}$$

so that if the initial time is t_o

$$t - t_o = \int dt = \int \frac{\pm\, dr}{\sqrt{2(\frac{\mu}{r} + C)}} =$$

$$\pm \frac{3C^2 r^2 - 4C\mu r + 8\mu^2}{15C^3} \sqrt{2\mu + Cr} + C_1 \tag{74}$$

if the velocity does not change sign between t_o and t. Let the initial position vector $\vec{r}_o$ have components

$$x_o^1 = r_o \cos \phi \cos \theta$$

$$x_o^2 = r_o \cos \phi \sin \theta$$

$$x_o^3 = r_o \sin \phi \tag{75}$$

Let the initial velocity vector have magnitude v_o. Then by (73)

$$C = \frac{1}{2} v_o^2 - \frac{\mu}{r_o} \tag{76}$$

and by (73)

$$C_1 = \mp \frac{3C^2 r_o^2 - 4C\mu r_o + 8\mu^2}{15C^3} \sqrt{2\mu + Cr} \tag{77}$$

Let us take as orbital elements in the straight line case $(r_o,\ e = \infty,\ \theta,\ \phi,\ v_o,\ t_o)$. Then using the above formulas we can determined $(x^1, x^2, x^3, \dot{x}^1, \dot{x}^2, \dot{x}^3)$ at time t from the orbital elements. Conversely it is trivial to determine the orbital elements from the position and velocity at time t_o.

The above formulas should be put in better form for computations, but since we will really never be concerned with the straight line case, we shall not bother to do so.

IV. DETERMINATION OF CONIC SECTION ORBITAL ELEMENTS GIVEN THE POSITION AND VELOCITY

Suppose we are given the position and velocity $(\vec{r}_o, \vec{v}_o) = (x_o^1, x_o^2, x_o^3, \dot{x}_o^1, \dot{x}_o^2, \dot{x}_o^3)$ at time t_o and wish to determine the conic section elements $(p, e, I, \Omega, \omega, t_p)$ as defined in (2) and (3).

We first calculate the angular momentum vector as given in (13),

$$\vec{G} = \vec{r}_o \times \vec{v}_o \tag{78}$$

If $G = |\vec{G}| = 0$, we have the straight line case for which the determination of the elements follows easily from the discussion in Section III-D. If $G \neq 0$, we determine the semi-latus rectum p from (22).

By (29) we have

$$\frac{1}{a} = \frac{2}{r} - \frac{v_o^2}{\mu} \tag{79}$$

and by (27) we have

$$e = +\sqrt{1 - \frac{p}{a}} \tag{80}$$

The inclination I and ascending node Ω are determined by (15) and (16). If $I = 0^o$ or 180^o, Ω is indeterminant so we can take $\Omega = 0$ by convention in this case.

Solving (30) and (40) for b_{jk}, we have

$$\left.\begin{aligned} b_{j1} &= \frac{1}{y_o^1 \dot{y}_o^2 - \dot{y}_o^1 y_o^2}\left(x^j \dot{y}^2 - \dot{x}^j y^2\right) \\ b_{j2} &= \frac{1}{y_o^1 \dot{y}_o^2 - \dot{y}_o^1 y_o^2}\left(\dot{x}^j y^1 - x^j \dot{y}^1\right) \end{aligned}\right\} \quad j = 1,2,3 \tag{81}$$

By (31), we have for $j = 3$,

$$\sin \omega = \frac{b_{31}}{\sin I}$$

$$\cos \omega = \frac{b_{32}}{\sin I} \tag{82}$$

If $I = 0^\circ$ or 180° so that by convention $\Omega = 0$, we can use

$$\sin \omega = - b_{12} = \pm b_{21}$$

$$\cos \omega = b_{11} = \pm b_{22} \tag{83}$$

where the $+$ sign of the $\pm$ sign is to be used if $I = 0$ and the $-$ sign is to be used if $I = 180^\circ$.

In order to determine the time of perigee crossing t_p and the components of position and velocity in the orbital plane y^1, y^2, $\dot{y}^1$, $\dot{y}^2$ for determining the argument of perigee ω from (81), (82) and (83), we must consider three separate cases.

IV-A. Parabolic Motion

If $e = 1$ or if $|e - 1| < \epsilon$, where ϵ is an accuracy constant depending on the number of places in floating point computations on the computer being used, then we are in the parabolic case and would set

$e = 1$ identically. Equation (35) is

$$r = \frac{p}{1 + \cos \psi} \tag{84}$$

and (33) with $a = \infty$ is

$$r \frac{dr}{dt} = \sqrt{2\mu r - \mu p}$$

Differentiating (84) and using (34) we have

$$\frac{dr}{dt} = \sqrt{\frac{\mu}{p}} \sin \psi$$

We thus have at the initial time t_o

$$\cos \psi_o = \frac{p}{r_o} - 1$$

$$\sin \psi_o = \frac{\sqrt{2pr_o - p^2}}{r_o} \tag{85}$$

from which ψ_o is easily determined with $-\pi < \psi_o < \pi$. By (37) and (38) the time t_p of perigee passage is then

$$t_p = t_o - \frac{1}{2n} \left(\tan \frac{\psi_o}{2} + \frac{1}{3} \tan^3 \frac{\psi_o}{2} \right) \tag{86}$$

where n is given by (36). The coordinates $y_o^1, y_o^2, \dot{y}_o^1, \dot{y}_o^2$ are given by (32) and (41) for use in determining ω.

IV-B. Elliptic Motion

We now suppose that $0 \leq e < 1$ and $a > 0$. If $e < \epsilon$, where ϵ is an accuracy constant, we set $e = 0$ identically for the circular case.

If $e = 0$, we would take $t_p = t_o$ and for use in determining ω assume that

$$y_o^1 = r_o$$

$$y_o^2 = 0$$

$$\dot{y}_o^1 = 0$$

$$\dot{y}_o^2 = v_o \tag{87}$$

If $e \neq 0$, we differentiate (46) with respect to time to obtain

$$\frac{dr}{dt} = ae \sin \xi \frac{d\xi}{dt} = \frac{na^2 e \sin \xi}{r} \tag{88}$$

by (56). Since

$$\frac{dr}{dt} = \frac{\vec{r} \cdot \vec{v}}{r} \tag{89}$$

this gives at the initial time

$$\sin \xi = \frac{\vec{r}_o \cdot \vec{v}_o}{na^2 e} \tag{90}$$

Further, (46) can be put in the form

$$\cos \xi = \frac{1}{e}(1 - \frac{r_o}{a}) \tag{91}$$

Then the time t_p of perigee passage is given by (47) in the form

$$t_p = t_o - \frac{1}{n}(\xi - e \sin \xi) \tag{92}$$

and y_o^1, y_o^2, $\dot{y}_o^1$, $\dot{y}_o^2$ are given by (52), (53), (57), (58) for use in determining ω.

IV-C. Hyperbolic Motion

We now suppose that $e > 1$ and $a < 0$. Differentiating (62) with respect to time we obtain

$$\frac{dr}{dt} = |a|\ e \sinh \xi \frac{d\xi}{dt} = \frac{n|a|^2 e \sinh \xi}{r} \tag{93}$$

by (70). Using (89), we have at the initial time

$$\sinh \xi = \frac{\vec{r}_o \cdot \vec{v}_o}{n|a|^2 e} \tag{94}$$

Further, (62) can be put in the form

$$\cosh \xi = \frac{1}{e}\left(1 + \frac{r_o}{a}\right) \tag{95}$$

Then the time t_p of perigee passage is given by (63) in the form

$$t_p = t_o - \frac{1}{n}(e \sinh \xi - \xi) \tag{96}$$

and y_o^1, y_o^2, $\dot{y}_o^1$, $\dot{y}_o^2$ are given by (67), (68), (71), (72) for use in determining ω.

V. LAMBERT PROBLEM FOR THE TIME OF TRAVEL BETWEEN TWO POINTS IN A CONIC SECTION ORBIT

Lambert's problem is to determine the elements $(p, e, I, \Omega, \omega, t_p)$ of the conic section orbit which has position vectors $\vec{r}_1$ at time t_1 and $\vec{r}_2$ at time t_2 with $t_1 < t_2$. We shall consider a variation on the problem in that we shall take

$$\vec{r}_1,\ v_1 = |\vec{v}_1|,\ t_1,\ \vec{r}_2 \tag{97}$$

as given and determine the elements $(p, e, I, \Omega, \omega, t_p)$ and the time of flight $(t_2 - t_1)$ from $\vec{r}_1$ to $\vec{r}_2$.

If $\vec{r}_2$ and $\vec{r}_1$ are parallel ($\vec{r}_2 = k\vec{r}_1$ with $k \geq 0$), then we have the case of straight line motion with elements easily determined by the formulas in section III-D. If $\vec{r}_1$ and $\vec{r}_2$ are anti-parallel ($\vec{r}_2 = k\vec{r}_1$ with $k < 0$), then the orbital plane is indeterminant and there are infinitely many orbits (all with the same elements p, e, t_p, $t_2 - t_1$) satisfying the conditions (97).

Now let us suppose $\vec{r}_1 \times \vec{r}_2 \neq \vec{0}$. The angle θ from $\vec{r}_1$ to $\vec{r}_2$ rotating in the direction of motion is then determined by

$$\left.\begin{aligned} \sin\theta &= \pm \frac{|\vec{r}_1 \times \vec{r}_2|}{|\vec{r}_1|\,|\vec{r}_2|} \\ \cos\theta &= \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1|\,|\vec{r}_2|} \end{aligned}\right\} \tag{98}$$

where the $+$ sign is used if $0 < \theta < 2\pi$ and the $-$ sign is used if $\pi < \theta < 2\pi$. Since we are concerned with rapid intercept orbits, we shall assume $0 < \theta < \pi$ in the program, although for the sake of completeness we shall allow both cases in the formulas that follow. The distance c between the points $\vec{r}_1$ and $\vec{r}_2$ is given by

$$c^2 = r_1^{\,2} + r_2^{\,2} - 2\vec{r}_1 \cdot \vec{r}_2 \tag{99}$$

The unit vector normal to the orbital plane is

$$\hat{G} = \pm \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \tag{100}$$

where the $+$ sign is used if $0 < \theta < \pi$ and the $-$ sign is used if $\pi < \theta \leq 2\pi$. The inclination I and ascending node Ω are determined from $\hat{G}$ by formulas analogeous to (15) and (16).

By (29), the semi-major axis a is

$$\frac{1}{a} = \frac{2}{r_1} - \frac{v_1^2}{\mu} \tag{101}$$

and the unknown magnitude of velocity v_2 is

$$v_2^2 = \mu\left(\frac{2}{r_2} - \frac{1}{a}\right) \tag{102}$$

If the right side of (102) is negative or zero, a conic section orbit satisfying (97) does not exist. By (29) and (33) we have

$$\dot{r}_i^2 + \frac{\mu p}{r_i^2} = v_i^2 \quad , \quad i = 1,2 \tag{103}$$

Now consider unit vectors $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$ in the orbital plane with $\vec{\varepsilon}_1$ pointing towards perigee and with $\vec{\varepsilon}_2$ normal to $\vec{\varepsilon}_1$ pointing in the direction of motion at perigee. We have

$$\vec{r}_i = y_i^1 \vec{\varepsilon}_1 + y_i^2 \vec{\varepsilon}_2 \quad , \quad i = 1,2 \tag{104}$$

If (x_i^1, x_i^2, x_i^3) are the components of $\vec{r}_i$ in the usual coordinate system, (30) gives

$$x_i^j = \sum_{k=1}^{2} b_{jk}\, y_i^k \quad , \quad i = 1,2 \;;\; j = 1,2,3 \tag{105}$$

Solving for b_{jk}, we obtain

$$\left.\begin{aligned} b_{j1} &= \frac{1}{y_1^1 y_2^2 - y_2^1 y_1^2}\left(x_1^j y_2^2 - x_2^j y_1^2\right) \\ b_{j2} &= \frac{1}{y_1^1 y_2^2 - y_2^1 y_1^2}\left(x_2^j y_1^1 - x_1^j y_2^1\right) \end{aligned}\right\} j = 1,2,3 \tag{106}$$

The argument of perigee ω could then be determined by (82) and (83) if y_i^1, y_i^2 were known.

To recapitulate, we are given values of the parameters (97) from which we determined I, Ω, a and v_2. In order to determine (p, e, ω, t_p, t_2) we shall have to consider separately the cases of parabolic motion $(a = \infty)$, elliptic motion $(a > 0)$ and hyperbolic motion $(a < 0)$ making use of relations (98) through (106).

V-A. Parabolic Motion

If $a = \infty$ so that $e = 1$, we have by (35)

$$r_i = \frac{p}{1 + \cos \psi_i} = \frac{p}{2} \sec^2 \frac{\psi_i}{2} \quad , \quad i = 1,2 \tag{107}$$

Then by (34) and (107)

$$\dot{r}_i = \sqrt{\frac{\mu}{p}} \sin \psi_i \quad , \quad i = 1,2 \tag{108}$$

so that by (107) and (108)

$$\left.\begin{aligned} \cos \psi_i &= \frac{p}{r_i} - 1 \\ \sin \psi_i &= \sqrt{\frac{p}{\mu}}\ \dot{r}_i \end{aligned}\right\} \quad i = 1,2 \tag{109}$$

From (37) and (38) we obtain

$$n(t_i - t_p) = \frac{1}{2}\left(\tan \frac{\psi_i}{2} + \frac{1}{3} \tan^3 \frac{\psi_i}{2}\right) \quad , \quad i = 1,2 \tag{110}$$

with

$$\theta = \psi_2 - \psi_1 \tag{111}$$

By (32), equations (104) take the form

$$\vec{r}_i = r_i \cos \psi_i \, \vec{\varepsilon}_1 + r_i \sin \psi_i \, \vec{\varepsilon}_2 \quad , \qquad i = 1,2$$

so that by (99)

$$\begin{aligned} c^2 &= r_1^{\,2} + r_2^{\,2} - 2 r_1 r_2 \cos(\psi_2 - \psi_1) \\ &= (r_1 + \Omega_2)^2 - 2 r_1 \Omega_2 \cos^2\left(\frac{\psi_2 - \psi_1}{2}\right) \end{aligned}$$

which implies

$$2\sqrt{r_1 r_2} \cos\left(\frac{\psi_2 - \psi_1}{2}\right) = \pm \sqrt{(r_1 + r_2 + c)\,(r_1 + r_2 - c)} \tag{112}$$

where the + sign is used if $\theta = \psi_2 - \psi_1 < \pi$ and the $-$ sign is used if $\theta = \psi_2 - \psi_1 > \pi$. Substituting (107) into the left side of (112) we obtain

$$1 + \tan\frac{\psi_1}{2} \tan\frac{\psi_2}{2} = \pm \frac{\sqrt{(r_1 + r_2 + c)\,(r_1 + r_2 - c)}}{p} \tag{113}$$

It also follows from (107) that

$$r_1 + r_2 = \frac{p}{2}\left(2 + \tan^2\frac{\psi_1}{2} + \tan^2\frac{\psi_2}{2}\right)$$

The last two equations give

$$(r_1 + r_2 + c) + (r_1 + r_2 - c) \mp 2\sqrt{(r_1 + r_2 + c)\,(r_1 + r_2 - c)}$$

$$= \left(\tan\frac{\psi_2}{2} - \tan\frac{\psi_1}{2}\right)^2$$

which implies

$$\frac{\sqrt{r_1 + r_2 + c} \mp \sqrt{r_1 + r_2 - c}}{\sqrt{p}} = \tan\frac{\psi_2}{2} - \tan\frac{\psi_1}{2} \tag{114}$$

Subtracting the equation of (110) with $i = 1$ from the one with $i = 2$, we obtain

$$6n(t_2 - t_1) = \left(\tan\frac{\psi_2}{2} - \tan\frac{\psi_1}{2}\right)\left[3\left(1 + \tan\frac{\psi_1}{2}\tan\frac{\psi_2}{2}\right) + \left(\tan\frac{\psi_2}{2} - \tan\frac{\psi_1}{2}\right)^2\right] \tag{115}$$

Equations (36), (113), (114) and (115) yield

$$6\mu^{1/2}(t_2 - t_1) = (r_1 + r_2 + c)^{3/2} \mp (r_1 + r_2 - c)^{3/2} \tag{116}$$

where the $-$ sign is used if $0 < \theta < \pi$ and the $+$ sign is used if $\pi < \theta < 2\pi$. This result, determining t_2 in terms of t_1, is called Euler's equation for parabolic motion and is given in Ref. 3, pp. 157-158.

By (109) we have

$$\tan\frac{\psi_i}{2} = \frac{\sin\psi_i}{1 + \cos\psi_i} = \frac{r_i\dot{r}_i}{\sqrt{\mu p}} \tag{117}$$

so that (114) becomes

$$r_2\dot{r}_2 - r_1\dot{r}_1 = \sqrt{\mu}\left[\sqrt{r_1 + r_2 + c} \mp \sqrt{r_1 + r_2 - c}\right] \tag{118}$$

Equations (111), (113), (117) and (118) yield

$$\tan\frac{\theta}{2} = \tan\left(\frac{\psi_2 - \psi_1}{2}\right) = \frac{\tan\frac{\psi_2}{2} - \tan\frac{\psi_1}{2}}{1 + \tan\frac{\psi_2}{2}\tan\frac{\psi_1}{2}}$$

$$= \sqrt{p}\left[\frac{\pm\sqrt{r_1 + r_2 + c} - \sqrt{r_1 + r_2 - c}}{\sqrt{(r_1 + r_2 + c)(r_1 + r_2 - c)}}\right] \tag{119}$$

where the + sign of the $\pm$ is used if $0 < \theta < \pi$ and the $-$ sign is used if $\pi < \theta < 2\pi$. All the quantities in (119) except p are known, so that (119) determines p. Then $\dot{r}_1$, $\dot{r}_2$ are determined by (103) in the form

$$\dot{r}_i = (\pm)_i\sqrt{v_i^2 - \frac{\mu p}{r_i^2}} = (\pm)_i\sqrt{\frac{2\mu}{r_i} - \frac{\mu p}{r_i^2}}, \quad i = 1,2 \tag{120}$$

where the + sign is used if $t_i < t_p$ and the $-$ sign is used if $t_i > t_p$. However, whether t_i is before or after the time of perigee t_p is not yet known, so $\dot{r}_i$ is determined only up to sign. The tangent of $\frac{\psi_i}{2}$ is given by (117) up to sign and the value of $(t_i - t_p)$ is given by (110) up to sign. We know that

$$(\pm)_2\left|\tan\frac{\psi_2}{2} + \frac{1}{3}\tan^3\frac{\psi_2}{2}\right| - (\pm)_1\left|\tan\frac{\psi_1}{2} + \frac{1}{3}\tan^3\frac{\psi_1}{2}\right| = 2n(t_2 - t_1) \tag{121}$$

where the right side is known from (116). In general there is only one combination of signs which will make (121) valid, so we may regard the sign ambiguity for $\dot{r}_i$ and $\tan\frac{\psi_i}{2}$ as resolved. Then t_p is determined by (110), $\cos\psi_i$ and $\sin\psi_i$ by (109), y_i^j by (32) and ω by (106), (82) and (83).

V-B. Elliptic Motion

If $a > 0$ so that $0 \leq e < 1$ we have by (46)

$$r_i = a(1 - e \cos \xi_i) \quad , \quad i = 1,2 \tag{122}$$

Then by (56) and (122)

$$\dot{r}_i = \frac{\sqrt{\mu a}\; e \sin \xi_i}{r_i} \quad , \quad i = 1,2 \tag{123}$$

Equations (122) and (123) can be written

$$\left.\begin{aligned} e \cos \xi_i &= 1 - \frac{r_i}{a} \\ e \sin \xi_i &= \frac{r_i \dot{r}_i}{\sqrt{\mu a}} \end{aligned}\right\} \quad i = 1,2 \tag{124}$$

Squaring these equations and adding we obtain

$$e^2 = \left(1 - \frac{r_i}{a}\right)^2 + \frac{r_i^2 \dot{r}_i^2}{\mu a} \quad , \quad i = 1,2 \tag{125}$$

Kepler's equation (47) implies

$$n(t_i - t_p) = \xi_i - e \sin \xi_i \quad , \quad i = 1,2 \tag{126}$$

By (52) and (53) equations (104) take the form

$$\vec{r}_i = a(\cos \xi_i - e)\, \vec{\epsilon}_1 + a\sqrt{1 - e^2}\, \sin \xi_i\, \vec{\epsilon}_2 \quad , \quad i = 2$$

so that by (99)

$$c^2 = a^2(\cos \xi_2 - \cos \xi_1)^2 + a^2(1 - e^2)(\sin \xi_2 - \sin \xi_1)^2$$

$$= 4a^2 \left[1 - e^2 \cos^2 \frac{(\xi_1 + \xi_2)}{2}\right] \sin^2 \frac{(\xi_2 - \xi_1)}{2} \tag{127}$$

It follows from (122) that

$$r_1 + r_2 = 2a \left[1 - e \cos \frac{(\xi_1 + \xi_2)}{2} \cos \frac{(\xi_2 - \xi_1)}{2}\right] \tag{128}$$

Subtracting the equation of (126) with $i = 1$ from the one with $i = 2$ we obtain

$$n(t_2 - t_1) = (\xi_2 - \xi_1) - 2 e \cos \frac{(\xi_1 + \xi_2)}{2} \sin \frac{(\xi_2 - \xi_1)}{2} \tag{129}$$

Continuing to follow Ref. 7, we define the angles α and β by

$$\left.\begin{aligned} \cos \frac{(\alpha + \beta)}{2} &= e \cos \frac{(\xi_1 + \xi_2)}{2} \qquad & 0 \le \alpha + \beta < 2\pi \\ \alpha - \beta &= \xi_2 - \xi_1 - 2m\pi \qquad & 0 \le \alpha - \beta < 2\pi \end{aligned}\right\} \tag{130}$$

where m is the number of complete circuits made between times t_1 and t_2. Inequalities (130) imply

$$0 \le \alpha < 2\pi \quad , \quad -\pi \le \beta < \pi \tag{131}$$

Equations (127), (128) and (129) become

$$\frac{c}{2a} = \sin \frac{(\alpha + \beta)}{2} \sin \frac{(\alpha - \beta)}{2} \tag{132}$$

$$\frac{r_1 + r_2}{2a} = 1 - \cos\frac{(\alpha + \beta)}{2}\cos\frac{(\alpha - \beta)}{2} \tag{133}$$

$$n(t_2 - t_1) = 2m\pi + \alpha - \beta - 2\cos\frac{(\alpha + \beta)}{2}\sin\frac{(\alpha - \beta)}{2} \tag{134}$$

There is no ambiguous sign in (132) because by (130)

$$0 \leq \frac{\alpha + \beta}{2} < \pi \quad , \quad \sin\frac{(\alpha + \beta)}{2} \geq 0$$

$$0 \leq \frac{\alpha - \beta}{2} < \pi \quad , \quad \sin\frac{(\alpha - \beta)}{2} \geq 0 \tag{135}$$

Equations (132) and (133) imply

$$\sin\frac{\alpha}{2} = \frac{+\sqrt{r_1 + r_2 + c}}{2\sqrt{a}} \tag{136}$$

$$\sin\frac{\beta}{2} = \frac{\pm\sqrt{r_1 + r_2 - c}}{2\sqrt{a}} \tag{137}$$

so that α and β are determined in terms of the known quantities r_1, r_2, c and a, except for ambiguity in sign in (137). There is no ambiguity in (136) because of (131). If either the right side in (136) or that in (137) are greater than 1 in absolute value, then an elliptic orbit passing through the two given points with the given magnitude of velocity at the first one does not exist. Equation (134) can be written in the form

$$n(t_2 - t_1) = 2m\pi + (\alpha - \beta) - (\sin\alpha - \sin\beta) \tag{138}$$

which is called Lambert's theorem in Ref. 4, p. 51. We have thus determined the time t_2 except for the number m of revolutions and the ambiguity of sign in (137). For the rapid intercept problem of interest to us we can assume $m = 0$.

By (111) and (49) we have

$$\tan\frac{\theta}{2} = \tan\left(\frac{\psi_2 - \psi_1}{2}\right) = \frac{\tan\frac{\psi_2}{2} - \tan\frac{\psi_1}{2}}{1 + \tan\frac{\psi_2}{2}\tan\frac{\psi_1}{2}}$$

$$= \frac{\sqrt{1 - e^2}\left[\tan\frac{\xi_2}{2} - \tan\frac{\xi_1}{2}\right]}{(1 - e) + (1 + e)\tan\frac{\xi_2}{2}\tan\frac{\xi_1}{2}}$$

$$= \frac{\sqrt{1 - e^2}\left[\sin\frac{\xi_2}{2}\cos\frac{\xi_1}{2} - \sin\frac{\xi_1}{2}\cos\frac{\xi_2}{2}\right]}{(1 - e)\cos\frac{\xi_2}{2}\cos\frac{\xi_1}{2} + (1 + e)\sin\frac{\xi_2}{2}\sin\frac{\xi_1}{2}}$$

$$= \frac{\sqrt{1 - e^2}\sin\left(\frac{\xi_2 - \xi_1}{2}\right)}{\cos\left(\frac{\xi_2 - \xi_1}{2}\right) - e\cos\left(\frac{\xi_1 + \xi_2}{2}\right)}$$

By (130) this becomes

$$\tan\frac{\theta}{2} = \frac{\sqrt{1 - e^2}\sin\left(\frac{\alpha - \beta}{2}\right)}{\cos\left(\frac{\alpha - \beta}{2}\right) - \cos\left(\frac{\alpha + \beta}{2}\right)}$$

$$= \frac{\sqrt{1 - e^2}\sin\left(\frac{\alpha - \beta}{2}\right)}{2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}} \qquad (139)$$

Thus by (135), (136) and (139) we can resolve the sign ambiguity in (137) as follows:

(i) $0 < \theta < \pi \rightarrow \sin\frac{\beta}{2} > 0$

(ii) $\pi < \theta < 2\pi \rightarrow \sin\frac{\beta}{2} < 0$

We now calculate the remaining elliptic orbital elements. The eccentricity e is determined by solving (139), obtaining

$$e^2 = 1 - \frac{4 \tan^2 \frac{\theta}{2} \sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2}}{\sin^2 \left(\frac{\alpha - \beta}{2}\right)} \tag{140}$$

The semi-latus rectum p is given by (27). The quantities $\dot{r}_1$ and $\dot{r}_2$ are determined by solving (125), obtaining

$$\dot{r}_i = (\pm)_i \frac{\sqrt{\mu a}}{r_i} \sqrt{e^2 - \left(1 - \frac{r_i}{a}\right)^2} \quad , \quad i = 1,2 \tag{141}$$

where the ambiguous sign $(\pm)_i$ for $\dot{r}_i$ is to be resolved. The values of $\sin \xi_i$ (with ambiguous sign $(\pm)_i$) and $\cos \xi_i$ are given by (124) if $e > 0$. Subtracting the equation of (126) with $i = 1$ from the one with $i = 2$ and comparing with (130) and (138) we have

$$(\pm)_2 |\sin \xi_2| - (\pm)_1 |\sin \xi_1| = \frac{\sin \alpha - \sin \beta}{e} \tag{142}$$

In general there is only one combination of signs which will make (142) valid, so we may regard the sign ambiguity as removed.

Knowing $\sin \xi_i$ and $\cos \xi_i$ $(i = 1,2)$, we can determine ξ_1 and ξ_2 subject to the restriction provided by the second equation of (130) with $m = 0$ in the rapid intercept case. Then t_{p_2} is determined by either equation in (126). The quantities y_i^1 and y_i^2 $(i = 1,2)$ are determined by (52) and (53), and the argument of perigee ω is determined by (106), (82) and (83).

If $e = 0$ or $e < \epsilon$, where ϵ is an accuracy constant depending on the number of places in floating point computations on the computer being used, we are in the circular case with $r_1 = r_2$ and would set $e = 0$ identically. Since there is no perigee point, t_p is arbitrary, so we can

take $t_p = t_1$. We have

$$y_1^1 = r_1$$

$$y_1^2 = 0$$

$$y_2^1 = r_1 \cos\theta$$

$$y_2^2 = r_1 \sin\theta \tag{143}$$

and the argument of perigee ω is determined by (106), (82) and (83).

V-C. Hyperbolic Motion

If $a < 0$ so that $e > 1$ we have by (62)

$$r_i = |a|(e \cosh \xi_i - 1) \quad , \quad i = 1,2 \tag{144}$$

Then by (70) and (144)

$$\dot{r}_i = \frac{\sqrt{\mu |a|}\; e \sinh \xi_i}{r_i} \quad , \quad i = 1,2 \tag{145}$$

Equations (144) and (145) can be written

$$\left.\begin{aligned} e \cosh \xi_i &= 1 + \frac{r_i}{|a|} \\ e \sinh \xi_i &= \frac{r_i \dot{r}_i}{\sqrt{\mu |a|}} \end{aligned}\right\} \quad i = 1,2 \tag{146}$$

Squaring these equations and subtracting the second from the first we obtain

$$e^2 = \left(1 + \frac{r_i}{|a|}\right)^2 - \frac{r_i^2 \dot{r}_i^2}{\mu |a|} \quad , \quad i = 1,2 \tag{147}$$

Kepler's equation (63) implies

$$n(t_i - t_p) = e \sinh \xi_i - \xi_i \quad , \quad i = 1,2 \tag{148}$$

By (67) and (68) equations (104) take the form

$$\vec{r}_i = |a|\left(e - \cosh \xi_i\right)\vec{\varepsilon}_1 + |a|\sqrt{e^2 - 1}\, \sinh \xi_i\, \vec{\varepsilon} \quad , \quad i = 1,2 \tag{149}$$

so that by (99)

$$\begin{aligned} c^2 &= |a|^2\left(\cosh \xi_2 - \cosh \xi_1\right)^2 + |a|^2\left(e^2 - 1\right)\left(\sinh \xi_2 - \sinh \xi_1\right)^2 \\ &= -\,4|a|^2\left[1 - e^2 \cosh^2\left(\frac{\xi_1 + \xi_2}{2}\right)\right] \sinh^2\left(\frac{\xi_2 - \xi_1}{2}\right) \end{aligned} \tag{150}$$

It follows from (144) that

$$r_1 + r_2 = 2|a|\left[e \cosh\left(\frac{\xi_1 + \xi_2}{2}\right) \cosh\left(\frac{\xi_2 - \xi_1}{2}\right) - 1\right] \tag{151}$$

Subtracting the equation of (148) with $i = 1$ from the one with $i = 2$ we obtain

$$n\left(t_2 - t_1\right) = 2e \cosh\left(\frac{\xi_1 + \xi_2}{2}\right) \sinh\left(\frac{\xi_2 - \xi_1}{2}\right) - \left(\xi_2 - \xi_1\right) \tag{152}$$

We define quantities α and β by

$$\left.\begin{aligned} \cosh\left(\frac{\alpha+\beta}{2}\right) &= e \cosh\left(\frac{\xi_1+\xi_2}{2}\right) \qquad & \alpha+\beta > 0 \\ \alpha - \beta &= \xi_2 - \xi_1 \qquad & \alpha - \beta > 0 \end{aligned}\right\} \tag{153}$$

Inequalities (153) imply

$$\alpha > 0 \tag{154}$$

Equations (150), (151) and (152) become

$$\frac{c}{2|a|} = \sinh\left(\frac{\alpha+\beta}{2}\right) \sinh\left(\frac{\alpha-\beta}{2}\right) \tag{155}$$

$$\frac{r_1+r_2}{2|a|} = \cosh\left(\frac{\alpha+\beta}{2}\right)\cosh\left(\frac{\alpha-\beta}{2}\right) - 1 \tag{156}$$

$$n(t_2 - t_1) = 2\cosh\left(\frac{\alpha+\beta}{2}\right) \sinh\left(\frac{\alpha-\beta}{2}\right) - (\alpha - \beta) \tag{157}$$

There is no ambiguous sign in (155) because by the inequalities in (153), the hyperbolic sines are always positive. Equations (155) and (156) imply

$$\sinh\frac{\alpha}{2} = \frac{+\sqrt{r_1+r_2+c}}{2\sqrt{|a|}} \tag{158}$$

$$\sinh\frac{\beta}{2} = \frac{\pm\sqrt{r_1+r_2-c}}{2\sqrt{|a|}} \tag{159}$$

so that α and β are determined in terms of the known quantities r_1, r_2, c and a, except for the ambiguity in sign in (159). There is no ambiguity in (158) because of (154). Equation (157) can then be written in the form

$$n(t_2 - t_1) = (\sinh \alpha - \sinh \beta) - (\alpha - \beta) \tag{160}$$

We have thus determined the time t_2 except for the ambiguity of sign in (159).

By (111) and (65) we have

$$\tan \frac{\theta}{2} = \tan \frac{\psi_2 - \psi_1}{2} = \frac{\tan \frac{\psi_2}{2} - \tan \frac{\psi_1}{2}}{1 + \tan \frac{\psi_2}{2} \tan \frac{\psi_1}{2}}$$

$$= \frac{\sqrt{e^2 - 1} \left[\tanh \frac{\xi_2}{2} - \tanh \frac{\xi_1}{2}\right]}{(e - 1) + (e + 1) \tanh \frac{\xi_2}{2} \tanh \frac{\xi_1}{2}}$$

$$= \frac{\sqrt{e^2 - 1} \left[\sinh \frac{\xi_2}{2} \cosh \frac{\xi_1}{2} - \sinh \frac{\xi_1}{2} \cosh \frac{\xi_2}{2}\right]}{(e - 1) \cosh \frac{\xi_2}{2} \cosh \frac{\xi_1}{2} + (e + 1) \sinh \frac{\xi_2}{2} \sinh \frac{\xi_1}{2}}$$

$$= \frac{\sqrt{e^2 - 1} \sinh \left(\frac{\xi_2 - \xi_1}{2}\right)}{e \cosh \left(\frac{\xi_1 + \xi_2}{2}\right) - \cosh \left(\frac{\xi_2 - \xi_1}{2}\right)}$$

By (153) this becomes

$$\tan \frac{\theta}{2} = \frac{\sqrt{e^2 - 1} \sinh \left(\frac{\alpha - \beta}{2}\right)}{\cosh \left(\frac{\alpha + \beta}{2}\right) - \cosh \left(\frac{\alpha - \beta}{2}\right)}$$

$$= \frac{\sqrt{e^2 - 1} \sinh \left(\frac{\alpha - \beta}{2}\right)}{2 \sinh \frac{\alpha}{2} \sinh \frac{\beta}{2}} \tag{161}$$

Thus by the inequalities in (153) and (154) we can resolve the sign ambiguity in (159) as follows:

$$\text{(i)} \quad 0 < \theta < \pi \rightarrow \sinh \frac{\beta}{2} > 0$$

$$\text{(ii)} \quad \pi < \theta < 2\pi \rightarrow \sinh \frac{\beta}{2} < 0$$

We now calculate the remaining hyperbolic orbital elements. The eccentricity e is determined by solving (161), obtaining

$$e^2 = 1 + \frac{4 \tan^2 \frac{\theta}{2} \sinh^2 \frac{\alpha}{2} \sinh^2 \frac{\beta}{2}}{\sinh^2\left(\frac{\alpha - \beta}{2}\right)} \tag{162}$$

The semi-latus rectum p is given by (27). The quantities $\dot{r}_1$ and $\dot{r}_2$ are determined by solving (147), obtaining

$$\dot{r}_i = (\pm)_i \frac{\sqrt{\mu |a|}}{r_i} \sqrt{\left(1 + \frac{r_i}{|a|}\right)^2 - e^2} \quad , \quad i = 1,2 \tag{163}$$

where the ambiguous sign $(\pm)_i$ for $\dot{r}_i$ is to be resolved. The values of $\sinh \xi_i$ (with ambiguous sign $(\pm)_i$) and $\cosh \xi_i$ are given by (146). Subtracting the equation of (148) with $i = 1$ from the one with $= 2$ and comparing with (153) and (160) we have

$$(\pm)_2 |\sinh \xi_2| - (\pm)_1 |\sinh \xi_1| = \frac{\sinh \alpha - \sinh \beta}{e} \tag{164}$$

In general there is only one combination of signs which will make (164) valid, so we may regard the sign ambiguity as removed.

Knowing $\sinh \xi_i$ we can determine ξ_i. Then t_p is determined by either equation in (148). The quantities y_i^1 and y_i^2 $(i = 1,2)$ are determined by (67) and (68) and the argument of perigee ω is determined by (106), (82) and (83).

VI. LOGICAL FLOW OF THE COMPUTER PROGRAM TO CALCULATE TIME TO INTERCEPT VERSUS INITIAL VELOCITY

1. Initialize constants.

2. Read input data described in Section I; end program if no more data.

3. Calculate position and velocity of launching site at launch time (earth launch uses the formulas in Section II and parking orbit launch uses the formulas in Section III-B.).

4. For magnitude of velocity v_1 from initial magnitude VLNCH0 to final magnitude VLNCH1 at increments DVLNCH, calculate and store in arrays the following:

   a. Time to intercept the target satellite with given magnitude of launch velocity v_1 in the intercept orbit using the formulas in Section V; calculated iteratively because the target satellite is allowed to move in its orbit between launch epoch and intercept epoch.

   b. Velocity imparted at launch, which is equal to the vector velocity (of magnitude v_1) in the intercept orbit at launch calculated using the formulas in Section III minus the vector velocity of the launching site.

   c. Position and velocity in the intercept orbit at possible midcourse correction epochs calculated using the formulas in Section III.

5. Plot time to intercept versus magnitude of velocity imparted at launch. A point is deleted from the plot if the time of perigee in the intercept orbit is between the times of launch and intercept and if the perigee distance is less than the radius of the earth plus, say, 50 kilometers.

6. If midcourse corrections are to be considered, change the target satellite orbital elements by the input orbital element errors.

7. For each original intercept orbit calculated in (4.) and each midcourse correction epoch with midcourse launching position and velocity calculated in (4c.), do the following:

a. For magnitude of velocity v_1 from an initial to a final magnitude at given increments calculate and store in arrays the following:

i. Time to intercept the new target satellite with given magnitude of midcourse launch velocity v_1 in the intercept orbit using the formulas in Section V; calculated iteratively because the target satellite is allowed to move in its new orbit between midcourse correction epoch and intercept epoch.

ii. Velocity imparted at midcourse correction which is equal to the vector velocity (of magnitude v_1) in the new intercept orbit at the midcourse epoch calculated using the formulas in Section V minus the vector velocity in the original intercept orbit at the midcourse epoch.

b. Plot time from midcourse correction to intercept versus magnitude of velocity imparted at midcourse correction. Points are deleted which, by the criterion described in (5.), represent intercept orbits intersecting the earth's atmosphere.

8. Go to (2.).

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13. ABSTRACT

Formulas from Lambert's problem in celestial mechanics are presented for use in a computer program to calculate velocity requirements for rapid intercept of an earth satellite by a rocket fired from the earth or from an orbit about the earth. The rocket is assumed to receive an impulsive velocity at launch and the effect of impulsive midcourse corrections are considered. The effects of the earth rotation are included if the rocket is fired from the earth, but those due to the earth's atmosphere are ignored. The rocket and target satellite are assumed to be moving in conic section in the earth's gravitational field.

14. KEY WORDS

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